

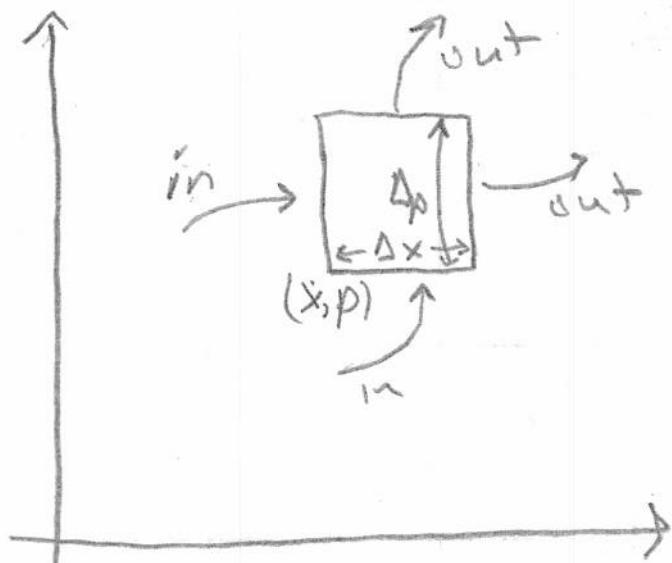
Evolution of the Distribution Function

Our simple model can only go so far. We now develop tools to help us deal with real particle distributions in phase space

Phase density

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$$n = \Psi(x, p, t) \Delta x \Delta p$$



$$n(t + \Delta t) = n(t) + (\text{flow in}) - (\text{flow out})$$

$$\text{flow in} = \Psi(x, p, t) \Delta p \times \Delta t$$

$$+ \Psi(x, p, t) \Delta x \cdot \dot{p} \Delta t$$

$$\text{flow out} = \Psi(x + \Delta x, p, t) \Delta p \cdot \dot{x} \Delta t$$

$$+ \Psi(x, p + \Delta x, t) \Delta x \cdot \dot{p} \Delta t$$

①

$$\rightarrow \frac{n(t+\Delta t) - n(t)}{\Delta t} = \psi(x, p, t) \Delta p \dot{x} + \psi(x, p, t) \Delta x \dot{p} - \psi(x+\Delta x, p, t) \Delta p \dot{x} - \psi(x, p+\Delta p, t) \Delta x \dot{p}$$

$$= - \dot{x} \Delta p \frac{\partial \psi}{\partial x} \Delta x$$

$$\rightarrow \dot{p} \Delta x \frac{\partial \psi}{\partial p} \Delta p$$

$$\rightarrow = \frac{\psi(x, p, t+\Delta t) - \psi(x, p, t)}{\Delta t} \Delta x \Delta p$$

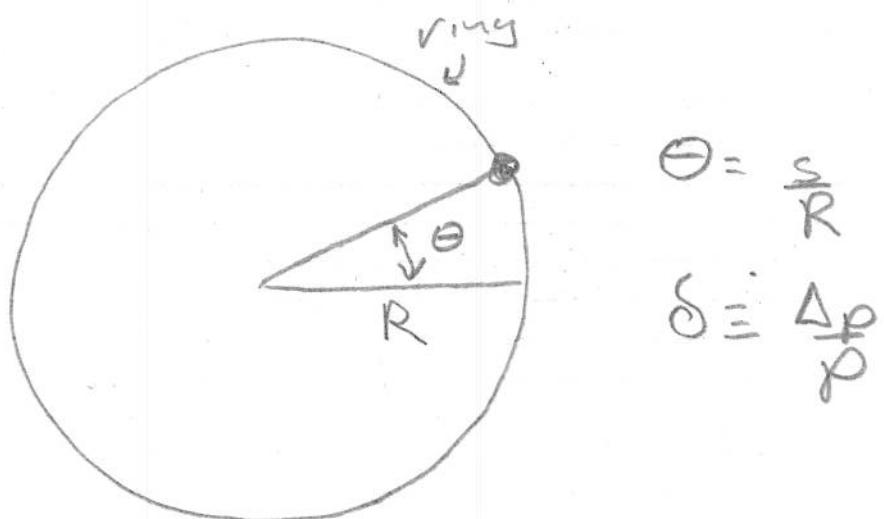
$$\rightarrow \boxed{\frac{\partial \psi}{\partial t} + \dot{x} \frac{\partial \psi}{\partial x} + \dot{p} \frac{\partial \psi}{\partial p} = 0}$$

Vlasov Equation

The Dispersion Relation

Apply a special case of
the Vlasov equation

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$$\Theta = \frac{\pi}{R}$$
$$\delta = \frac{\Delta p}{p_0}$$

The Vlasov equation becomes

$$\frac{\partial \Psi}{\partial t} + \dot{x} \frac{\partial \Psi}{\partial x} + \dot{p} \frac{\partial \Psi}{\partial p} = 0$$

$$\rightarrow x = R\Theta$$

$$\rightarrow \dot{x} = R\dot{\Theta}$$

$$\partial_x = R\partial\Theta$$

$$p = \delta p_0$$

$$\dot{p} = p_0 \dot{\delta}$$

$$\partial_p = p_0 \partial \delta$$

$$\rightarrow \frac{\partial \Psi}{\partial t} + \dot{\Theta} \frac{\partial \Psi}{\partial \Theta} + \dot{\delta} \frac{\partial \Psi}{\partial \delta} = 0$$

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$$\dot{\theta} = \omega$$

$$\dot{\delta} = \frac{1}{\beta^2} \frac{\Delta E}{E}$$

Recall that in our introduction to fine density perturbation, we found that if the current is described by

$$I = I_0 + I e^{i(\Omega t - n\theta)} \quad \begin{matrix} \text{frequency of} \\ \text{oscillation} \end{matrix} \quad \begin{matrix} \text{mode} \end{matrix}$$

then $\frac{dE}{dt} = (\text{energy lost per turn}) \left(\frac{\text{turns}}{\text{sec}} \right)$

$$= -e I_0 Z_{ll} e^{i(\Omega t - n\theta)} \left(\frac{\omega_0}{2\pi} \right) \quad \begin{matrix} \text{definition} \end{matrix}$$

$$\dot{\delta} = \frac{\dot{\Delta E}}{E} = -\frac{e\omega_0 I_0 Z_{ll}}{2\pi\beta^2 E} e^{i(\Omega t - n\theta)}$$

So just as we did with our currents, we define the density of a constant part and an oscillatory part

$$\Psi(\delta, \theta, t) = \Psi_0(\delta) + \Psi_1(\delta) e^{i(Rt-n\theta)}$$

Plug this into the Vlasov equation

$$\frac{\partial \Psi}{\partial t} = i R e^{i(Rt-n\theta)}$$

$$\dot{\theta} \frac{\partial \Psi}{\partial \theta} = -i w n e^{i(Rt-n\theta)}$$

$$i \frac{\partial \Psi}{\partial \delta} \approx - \frac{e \omega_0 I Z_{II}}{2 \pi \beta^2 E} e^{i(Rt-n\theta)} \frac{\partial \Psi_0}{\partial \delta}$$

$$\rightarrow i(\Omega - n\omega) \Psi_1 - \frac{\partial \Psi_0}{\partial \delta} \frac{e \omega_0 I Z_{II}}{2 \pi \beta^2 E} = 0$$

Convert $d\delta$ to $d\omega$

$$\omega T = 2\pi \rightarrow \frac{d\omega}{\omega} = -\frac{dT}{T} = -\delta \eta$$

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$$\rightarrow \frac{\partial}{\partial \delta} = -\omega_0 \eta \frac{\partial}{\partial \omega}$$

Combining

$$\Psi_i = \frac{ie\omega_0^2 I_1 Z_n \eta}{2\pi \beta^2 E} \frac{1}{(\Omega - n\omega)} \frac{\partial \Psi_0}{\partial \omega}$$

Integrate by $d\omega = -\omega_0 \eta d\delta$

$$\rightarrow \int \Psi_i(\omega) d\omega = -\omega_0 \eta \int \Psi_i(\delta) d\delta \\ = -\eta \frac{I_1}{e}$$

$$= \frac{ie\omega_0^2 I_1 Z_n \eta}{2\pi \beta^2 E} \int \frac{\left(\frac{\partial \Psi_0}{\partial \omega} \right)}{\Omega - n\omega} d\omega$$

$$\rightarrow \boxed{1 = -i \frac{e^2 \omega_0^2 Z_1}{2\pi \beta^2 E} \int \frac{\left(\frac{\partial \Psi_0}{\partial \omega} \right)}{\Omega - n\omega} d\omega}$$

Dispersion Relation

Application to the negative mass instability

Unbunched beam with $\delta = 0$

$$\rightarrow \Psi_0(\delta, \theta, t) = \frac{N}{2\pi} \delta(\delta)$$

If we write it in terms of ω we have

$$\Psi_0(\omega, \theta) = -\frac{N\eta\omega_0}{2\pi} \delta(\omega - \omega_0)$$

$$\rightarrow \int \frac{(\partial \Psi_0 / \partial \omega)}{(\Omega - n\omega)} d\omega = -\frac{N\eta\omega_0}{2\pi} \int \frac{\delta'(\omega - \omega_0)}{\Omega - n\omega} d\omega$$

remember $\int f(x) f'(x) dx = f'(x)$

$$\rightarrow = \frac{N\eta\omega_0}{2\pi} \frac{n}{(\Omega - n\omega)^2}$$

$$1 = -\frac{i e^2 \omega_0^2 Z_{11}}{2\pi \beta^2 E} \frac{N\eta\omega_0}{2\pi} \frac{n}{(\Omega - n\omega)^2}$$

$$\rightarrow (\Omega - nw)^2 = - \frac{ie\gamma I_0 Z_0 w_0^2 n}{2\pi \beta^2 E}$$

which is exactly what we got earlier.

Now consider a more realistic beam

$$\begin{aligned} \Psi(\delta, \theta) &= \frac{N}{2\pi} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\delta^2/2\sigma^2} \\ &= \frac{N}{(2\pi)^{\frac{3}{2}}\sigma} e^{-\delta^2/2\sigma^2} \end{aligned}$$

In terms of angular frequency

$$\Psi_0(\omega) = \frac{N}{(2\pi)^{\frac{3}{2}}\sigma} e^{-\frac{(\omega-\omega_0)^2}{2(\sigma\omega_0)^2}}$$

$\delta \rightarrow -\frac{(\omega-\omega_0)}{\sigma\omega_0}$

The dispersion integral becomes

$$\int_{-\infty}^{\infty} \frac{\partial \Psi_0}{\partial \omega} \frac{1}{\Omega - nw} d\omega$$

$$= - \frac{N}{(2\pi)^{\frac{3}{2}}\sigma} \frac{1}{(\sigma\omega_0\sigma)^2}$$

$$\int_{-\infty}^{\infty} \frac{d\omega (\omega - \omega_0)}{(\Omega - nw)} e^{-\frac{(\omega-\omega_0)^2}{2(\sigma\omega_0)^2}}$$

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$$= \frac{N}{(2\pi)^{\frac{3}{2}} \eta \omega_0 \sigma^2 n} \int_{-\infty}^{\infty} \frac{u}{u-u_0} e^{-\frac{u^2}{2}} du$$

where : $u = \frac{\omega - \omega_0}{\eta \omega_0 \sigma}$

$$u_0 = \frac{\Omega - n\omega_0}{\eta \omega_0 \sigma} = \frac{\Delta \Omega}{\eta \omega_0 \sigma n}$$

$$\rightarrow I = -i \frac{e^2 \omega_0^2 Z_1}{2\pi \beta^2 E} \frac{N}{(2\pi)^{\frac{3}{2}} \eta \omega_0 \sigma^2 n} \int_{-\infty}^{\infty} \frac{u}{u-u_0} e^{-\frac{u^2}{2}} du$$

$\downarrow \frac{e \omega_0 N}{2\pi} = I_0$

$$= -i \frac{e I_0 Z_1}{2\pi \beta^2 E \eta \sigma^2 n} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{u}{u-u_0} e^{-\frac{u^2}{2}} du \right]$$

$\curvearrowleft \equiv I_0(u_0)$ dispersion integral

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$$\text{Recall } \psi = \psi_0 + \psi_1 e^{i(\Omega t - n\theta)}$$

$$\hookrightarrow e^{i((\Delta\Omega + nw_0)t - n\theta)}$$

$$= e^{i[\Delta\Omega t + n(w_0 t - \theta)]}$$

If $\Delta\Omega$ has a negative imaginary part, then motion will be unstable. Use trick

$$\frac{1}{u-u_0} = -i \int_0^\infty e^{i(u-u_0)\alpha} d\alpha$$

$$\rightarrow I_D = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{u}{u-u_0} e^{-u^2/2} du$$

$$= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-u^2/2} \left(\int_0^\infty e^{i(u-u_0)\alpha} d\alpha \right) du$$

$$= \frac{-i}{\sqrt{2\pi}} \int_0^\infty e^{-iu\alpha} \int_{-\infty}^{\infty} u e^{-(u^2 - 2iu\alpha - \alpha^2)/2} e^{-\alpha^2/2} du d\alpha$$

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$$= \frac{-i}{\sqrt{2\pi}} \int_0^\infty e^{-iu_0\alpha} e^{-\alpha^2/2} \left(\int_{-\infty}^\infty u e^{-(u-i\alpha)^2/2} du \right) d\alpha$$

$$= \sqrt{2\pi} i \alpha$$

$$= \int_0^\infty a e^{-iu_0\alpha} e^{-\alpha^2/2} d\alpha$$

↑
If $\text{Im}(u_0) < 0$, then $I_0 < 1$

What's the point?

$$u_0 = \frac{\Delta \Omega}{\gamma w_0 \sigma_J}$$

If $u_0 = 0$, then $I_0 = 1$. If $(\text{Im}(u_0)) < 0$, then $e^{-iu_0\alpha} = e^{-i \text{Re } u_0 \alpha} \times e^{+i \text{Im } u_0 \alpha}$

will decay and $I_0(u_0) < 1$

From the dispersion relation

$$I = -i \frac{e I_0 Z_1}{2\pi \beta^2 E \gamma \sigma^2 n} I_0(u_0)$$

So this solution can only exist if

$$\left| \frac{e I_0 Z_{\parallel}}{2\pi\beta^2 E D \sigma^2 n} \right| > 1$$

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So motion will be stable if

$$\sigma^2 > \frac{e I_0}{2\pi\gamma\beta^2 E} \left| \frac{Z_{\parallel}}{n} \right|$$

↑ = RMS of momentum distribution

More generally, motion will be stable if

$$\left| \frac{Z_{\parallel}}{n} \right| < \frac{\alpha}{2} \frac{2\pi\gamma\beta^2 E \sigma^2}{e I_0}$$

↑ form factor which depends on type of distribution

→ "Keil-Schnell criterion"